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HOMOTOPICALLY EQUIVALENT SIMPLE LOOPS ON 2-BRIDGE SPHERES IN 2-BRIDGE LINK COMPLEMENTS (I)

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Abstract. In this paper and its sequels, we give a necessary and sufficient condition for two essential simple loops on a 2-bridge sphere in a 2-bridge link complement to be homotopic in the link complement. This paper treats the case when the 2-bridge link is a $(2, p)$-torus link and its sequels will treat the remaining cases.

Contents

1. Introduction

Let K be a 2-bridge link in S^3 and let S be a 4-punctured sphere in $S^3 - K$ obtained from a 2-bridge sphere of K. In [4], we gave a complete characterization of those essential simple loops in S which are null-homotopic in $S^3 - K$,

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and by using the result, we described all upper-meridian-pair-preserving epimorphisms between 2-bridge link groups. The purpose of this paper and its sequels is to give a necessary and sufficient condition for two essential simple loops on S to be homotopic in $S^3 - K$. In this paper, we treat the case when the 2-bridge link is a $(2, p)$-torus link (Main Theorem 2.7). The remaining cases will be treated in the sequels of this paper, and these results will be used to show the existence of a variation of McShane's identity for 2-bridge links (cf. [11]). For an overview of this series of works, we refer the reader to the research announcement [5].

This paper is organized as follows. In Section 2, we recall basic facts concerning 2-bridge links, and describe the main result of this paper. In Section 3, we recall the upper presentation of a 2-bridge link group, and recall key facts established in [4] concerning the upper presentation. In Section 4, we establish a very strong structure theorem (Theorem 4.9 and Corollary 4.11) for the annular diagram which arises in the study of the conjugacy problem by using small cancellation theory. Sections 5 and 6 are devoted to the proof of the main result. In the final section, Section 7, we give a topological proof of the if part of the main theorem.

2. Main result

Consider the discrete group, H, of isometries of the Euclidean plane $\mathbb{R}^2$ generated by the π-rotations around the points in the lattice $\mathbb{Z}^2$. Set $(\boldsymbol{S}^2, \boldsymbol{P}) := (\mathbb{R}^2, \mathbb{Z}^2)/H$ and call it the *Conway sphere*. Then $\boldsymbol{S}^2$ is homeomorphic to the 2-sphere, and $\boldsymbol{P}$ consists of four points in $\boldsymbol{S}^2$. We also call $\boldsymbol{S}^2$ the Conway sphere. Let $\boldsymbol{S} := \boldsymbol{S}^2 - \boldsymbol{P}$ be the complementary 4-times punctured sphere. For each $r \in \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$, let α_r be the unoriented simple loop in $\boldsymbol{S}$ obtained as the projection of a line in $\mathbb{R}^2 - \mathbb{Z}^2$ of slope r. Then α_r is *essential* in $\boldsymbol{S}$, i.e., it does not bound a disk nor a once-punctured disk in $\boldsymbol{S}$. Conversely, any essential simple loop in $\boldsymbol{S}$ is isotopic to α_r for a unique $r \in \hat{\mathbb{Q}}$. Then r is called the *slope* of the simple loop. Similarly, any simple arc δ in $\boldsymbol{S}^2$ joining two different points in $\boldsymbol{P}$ such that $\delta \cap \boldsymbol{P} = \partial\delta$ is isotopic to the image of a line in $\mathbb{R}^2$ of some slope $r \in \hat{\mathbb{Q}}$ which intersects $\mathbb{Z}^2$. We call r the *slope* of δ.

A *trivial tangle* is a pair (B^3, t), where B^3 is a 3-ball and t is a union of two arcs properly embedded in B^3 which is parallel to a union of two mutually disjoint arcs in ∂B^3. By a *rational tangle*, we mean a trivial tangle (B^3, t) which is endowed with a homeomorphism from $\partial(B^3, t)$ to $(\boldsymbol{S}^2, \boldsymbol{P})$. Through the homeomorphism we identify the boundary of a rational tangle with the Conway sphere. Thus the slope of an essential simple loop in $\partial B^3 - t$ is defined.

We define the *slope* of a rational tangle to be the slope of an essential loop on $\partial B^3 - t$ which bounds a disk in B^3 separating the components of t. (Such a loop is unique up to isotopy on $\partial B^3 - t$ and is called a *meridian* of the rational tangle.) We denote a rational tangle of slope r by $(B^3, t(r))$. By van Kampen's theorem, the fundamental group $\pi_1(B^3 - t(r))$ is identified with the quotient $\pi_1(\boldsymbol{S})/\langle\langle \alpha_r \rangle\rangle$, where $\langle\langle \alpha_r \rangle\rangle$ denotes the normal closure.

For each $r \in \hat{\mathbb{Q}}$, the *2-bridge link $K(r)$ of slope r* is defined to be the sum of the rational tangles of slopes ∞ and r, namely, $(S^3, K(r))$ is obtained from $(B^3, t(\infty))$ and $(B^3, t(r))$ by identifying their boundaries through the identity map on the Conway sphere $(\boldsymbol{S}^2, \boldsymbol{P})$. (Recall that the boundaries of rational tangles are identified with the Conway sphere.) By van Kampen's theorem again, the *link group* $G(K(r)) := \pi_1(S^3 - K(r))$ is identified with $\pi_1(\boldsymbol{S})/\langle\langle \alpha_\infty, \alpha_r \rangle\rangle$. Note that $K(r)$ has one or two components according as the denominator of r is odd or even. We call $(B^3, t(\infty))$ and $(B^3, t(r))$, respectively, the *upper tangle* and *lower tangle* of the 2-bridge link.

Let $\mathcal{D}$ be the *Farey tessellation*, that is, the tessellation of the upper half space $\mathbb{H}^2$ by ideal triangles which are obtained from the ideal triangle with the ideal vertices $0, 1, \infty \in \hat{\mathbb{Q}}$ by repeated reflection in the edges. Then $\hat{\mathbb{Q}}$ is identified with the set of the ideal vertices of $\mathcal{D}$. For each $r \in \hat{\mathbb{Q}}$, let Γ_r be the group of automorphisms of $\mathcal{D}$ generated by reflections in the edges of $\mathcal{D}$ with an endpoint r. It should be noted that Γ_r is isomorphic to the infinite dihedral group and the region bounded by two adjacent edges of $\mathcal{D}$ with an endpoint r is a fundamental domain for the action of Γ_r on $\mathbb{H}^2$, by virtue of Poincare's fundamental polyhedron theorem (see, for example, [8]). Let $\hat{\Gamma}_r$ be the group generated by Γ_r and Γ_∞. When $r \in \mathbb{Q} - \mathbb{Z}$, $\hat{\Gamma}_r$ is equal to the free product $\Gamma_r * \Gamma_\infty$, having a fundamental domain shown in Figure 1. Otherwise, $\hat{\Gamma}_r$ is the group generated by the edges of $\mathcal{D}$ or Γ_∞ according as $r \in \mathbb{Z}$ or $r = \infty$.

The following simple observation was made in [7].

Proposition 2.1 ([7, Proposition 4.6])**.** *If two elements s and s' of $\hat{\mathbb{Q}}$ belong to the same orbit $\hat{\Gamma}_r$-orbit, then the unoriented loops α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$.*

Thus the following question naturally arises.

Question 2.2. *Is the converse to the above proposition valid? If not, when are two essential simple loops on a 2-bridge sphere in a 2-bridge link complement homotopic in the link complement?*

If $r = \infty$, then $G(K(\infty))$ is a rank 2 free group, and the result of Komori and Series [3, Theorem 1.2] is equivalent to the affirmative answer to the first

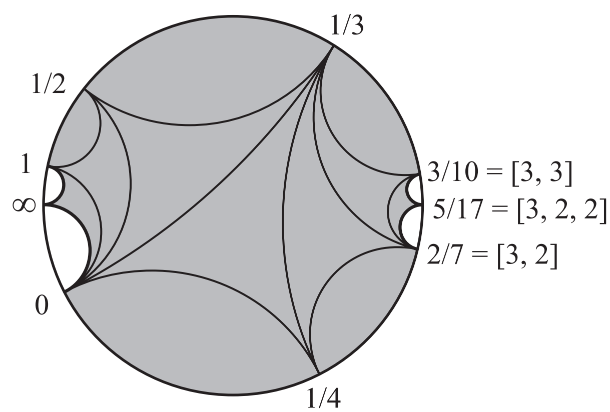


FIGURE 1. A fundamental domain of $\hat{\Gamma}_r$ in the Farey tessellation (the shaded domain) for $r = 5/17 = \cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{2}}} =: [3, 2, 2]$.

question in the above. In [4], we gave the following complete characterization of those essential simple loops on 2-bridge spheres of 2-bridge links which are null-homotopic in the link complements.

Theorem 2.3 ([4, Main Theorem 2.3]). *The loop α_s is null-homotopic in $S^3 - K(r)$ if and only if s belongs to the $\hat{\Gamma}_r$-orbit of ∞ or r.*

The purpose of this paper and its sequels is to solve the above question. In this paper, we give an answer to the question for the $(2, p)$-torus link $K(1/p)$ (Main Theorem 2.7).

In order to state the main result, we introduce a notation. For a rational number $r = 1/p$ ($p > 1$), let τ be the automorphism of the Farey tessellation defined as follows:

(i) If $p = 2p_0$ for some $p_0 \in \mathbb{Z}_+$, then τ is the reflection in the Farey edge $\langle 0, 1/p_0 \rangle$.

(ii) If $p = 2p_0 + 1$ for some $p_0 \in \mathbb{Z}_+$, then τ is the reflection in the geodesic with an endpoint 0 which bisects the Farey edge $\langle 1/p_0, 1/(p_0+1) \rangle$. Thus τ is the reflection in the geodesic with endpoints 0 and $2/(2p_0 + 1)$.

In both cases, τ is the reflection in the geodesic with endpoints 0 and $2/p$, and interchanges ∞ with $1/p$ (see Figure 7). Moreover, the action of τ on the vertex set $\hat{\mathbb{Q}}$ of the Farey tessellation is given by $\tau(c/d) = c/(cp - d)$

(see Lemma 7.1). In particular, if $\tau(q_1/p_1) = q_2/p_2$, where (p_i, q_i) is a pair of relatively prime positive integers, then $q_1 = q_2$ and $q_1/(p_1 + p_2) = 1/p$.

Set $\tilde{\Gamma}_{1/p} = \langle \hat{\Gamma}_{1/p}, \tau \rangle$, the group generated by $\hat{\Gamma}_{1/p}$ and τ. Then $\tilde{\Gamma}_{1/p}$ is a $\mathbb{Z}_2$-extension of $\hat{\Gamma}_{1/p}$. Now our main theorem is stated as follows.

Main Theorem 2.4. *Let $p \geq 2$ be an integer. Then, for two distinct $s, s' \in \hat{\mathbb{Q}}$, the unoriented loops α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(1/p)$ if and only if one of the following holds.*

(1) *s and s' lie in the same $\tilde{\Gamma}_{1/p}$-orbit.*

(2) *s and s' are contained in the union of $\tilde{\Gamma}_{1/p}$-orbits of ∞ and $1/p$. In this case, both α_s and $\alpha_{s'}$ are null-homotopic in $S^3 - K(1/p)$.*

In the following, we give a reformulation of the main theorem so that it is more suitable for the proof. Suppose that r is a rational number with $0 < r < 1$. Write

$$r = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \ddots + \cfrac{1}{m_k}}} =: [m_1, m_2, \dots, m_k],$$

where $k \geq 1$, $(m_1, \dots, m_k) \in (\mathbb{Z}_+)^k$, and $m_k \geq 2$. Recall that the region, R, bounded by a pair of Farey edges with an endpoint ∞ and a pair of Farey edges with an endpoint r forms a fundamental domain of the action of $\hat{\Gamma}_r$ on $\mathbb{H}^2$ (see Figure 1). Let $I_1(r)$ and $I_2(r)$ be the closed intervals in $\hat{\mathbb{R}}$ obtained as the intersection with $\hat{\mathbb{R}}$ of the closure of R. To be precise, $I_1(r) = [0, r_1]$ and $I_2(r) = [r_2, 1]$, where

$$r_1 = \begin{cases} [m_1, m_2, \dots, m_{k-1}] & \text{if } k \text{ is odd,} \\ [m_1, m_2, \dots, m_{k-1}, m_k - 1] & \text{if } k \text{ is even,} \end{cases}$$

$$r_2 = \begin{cases} [m_1, m_2, \dots, m_{k-1}, m_k - 1] & \text{if } k \text{ is odd,} \\ [m_1, m_2, \dots, m_{k-1}] & \text{if } k \text{ is even.} \end{cases}$$

If $r = 1/p$ $(p > 1)$, then $I_1(r)$ is degenerate to the singleton $\{0\}$. And if $r = (p-1)/p$ $(p > 1)$, then $I_2(r)$ is degenerate to the singleton $\{1\}$. Otherwise, $I_1(r)$ and $I_2(r)$ are non-degenerate intervals, and the union $I_1(r) \cup I_2(r)$ forms a fundamental domain of the action of $\hat{\Gamma}_r$ on the domain of discontinuity of $\hat{\Gamma}_r$, the complement in $\hat{\mathbb{R}}$ of the closure of $\hat{\Gamma}_r\{\infty, r\}$. (In the exceptional case $r = 1/p$ (resp. $(p-1)/p$), the rational number 0 (resp. 1) lies in the limit

set and $I_2(r)$ (resp. $I_1(r)$) is a fundamental domain of the action of $\hat{\Gamma}_r$ on the domain of discontinuity.)

Lemma 2.5 ([4, Lemma 7.1])**.** *Suppose $0 < r < 1$. Then, for any $s \in \hat{\mathbb{Q}}$, there is a unique rational number $s_0 \in I_1(r) \cup I_2(r) \cup \{\infty, r\}$ such that s is contained in the $\hat{\Gamma}_r$-orbit of s_0, and in particular, α_s is homotopic to α_{s_0} in $S^3 - K(r)$.*

Thus the only if part of Theorem 2.3 is equivalent to the following theorem, except for the the trivial knot $K(0)$ and the trivial 2-component link $K(\infty)$.

Theorem 2.6 ([4, Theorem 7.2])**.** *Suppose $0 < r < 1$. Then, for any rational number $s \in I_1(r) \cup I_2(r)$, α_s is not null-homotopic in $S^3 - K(r)$.*

We can now reformulate our main theorem.

Main Theorem 2.7. *Let $p \geq 2$ be an integer. Then, for two distinct rational numbers $s, s' \in I_1(1/p) \cup I_2(1/p)$, the unoriented loops α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(1/p)$ if and only if $s = q_1/p_1$ and $s' = q_2/p_2$ satisfy $q_1 = q_2$ and $q_1/(p_1 + p_2) = 1/p$, where (p_i, q_i) is a pair of relatively prime positive integers.*

We prove this main theorem by interpreting the situation in terms of combinatorial group theory. In other words, we prove that two words representing the free homotopy classes of α_s and $\alpha_{s'}$ are conjugate in the 2-bridge link group $G(K(1/p))$ if and only if s and s' satisfy the conditions given in the statement of the theorem. The key tool used in the proof is small cancellation theory, which is one of the representative geometric techniques in combinatorial group theory, applied to reduced annular diagrams over 2-bridge link groups. The proof of the if part is contained in Section 5, and the only if part in Section 6. The topological proof of the if part is also discussed in Section 7.

3. Preliminaries

In this section, we introduce the upper presentation of a 2-bridge link group, and recall key facts established in [4] concerning it.

To find a presentation of the 2-bridge link group $G(K(r))$ explicitly, let a and b, respectively, be the elements of $\pi_1(B^3 - t(\infty), x_0)$ represented by the oriented loops μ_1 and μ_2 based on x_0 as illustrated in Figure 2. Then $\{a, b\}$ forms the meridian pair of $\pi_1(B^3 - t(\infty))$, which is identified with the free group $F(a, b)$. Note that μ_i intersects the disk, δ_i, in B^3 bounded by a component of $t(\infty)$ and the essential arc, γ_i, on $\partial(B^3, t(\infty)) = (\boldsymbol{S}^2, \boldsymbol{P})$ of slope $1/0$, in Figure 2. Obtain a word u_r in $\{a, b\}$ by reading the intersection of the

(suitably oriented) loop α_r with $\gamma_1 \cup \gamma_2$, where a positive intersection with γ_1 (resp. γ_2) corresponds to a (resp. b). Then the cyclic word (u_r) represents the free homotopy class of the (oriented) loop α_r (see the preceding paragraph of Definition 3.3 for the precise definition of a cyclic word). It then follows that

$$\begin{aligned} G(K(r)) = \pi_1(S^3 - K(r)) &\cong \pi_1(B^3 - t(\infty))/\langle\langle \alpha_r \rangle\rangle \\ &\cong F(a,b)/\langle\langle u_r \rangle\rangle \cong \langle a, b \,|\, u_r \rangle. \end{aligned}$$

This one-relator presentation is called the *upper presentation* of $G(K(r))$ (see [2]). If $r \neq \infty$, then α_r intersects γ_1 and γ_2 alternately, and hence a and b appear in (u_r) alternately. It is known by [9, Proposition 1] that there is a nice formula to find u_r as follows (see [4, Remark 1] for a geometric picture).

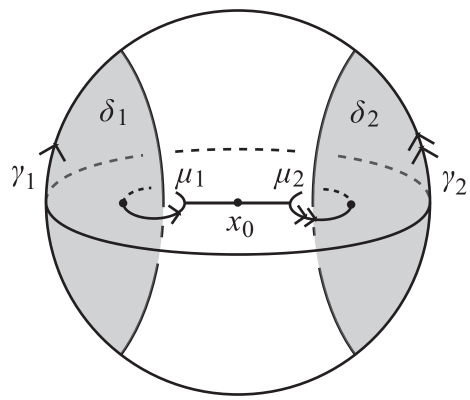


FIGURE 2. $\pi_1(B^3 - t(\infty), x_0) = F(a,b)$, where a and b are represented by μ_1 and μ_2, respectively.

Lemma 3.1. *Let p and q be relatively prime positive integers such that $p \geq 1$. For $1 \leq i \leq p-1$, let*

$$\epsilon_i = (-1)^{\lfloor iq/p \rfloor},$$

where $\lfloor x \rfloor$ is the greatest integer not exceeding x.

(1) *If p is odd, then*

$$u_{q/p} = a\hat{u}_{q/p} b^{(-1)^q} \hat{u}_{q/p}^{-1},$$

where $\hat{u}_{q/p} = b^{\epsilon_1} a^{\epsilon_2} \cdots b^{\epsilon_{p-2}} a^{\epsilon_{p-1}}$.

(2) *If p is even, then*

$$u_{q/p} = a\hat{u}_{q/p} a^{-1} \hat{u}_{q/p}^{-1},$$

where $\hat{u}_{q/p} = b^{\epsilon_1} a^{\epsilon_2} \cdots a^{\epsilon_{p-2}} b^{\epsilon_{p-1}}$.

Remark 3.2. For $r = 0/1$ and $r = 1/0$, we have $u_{0/1} = ab$ and $u_{1/0} = 1$.

We now define the sequence $S(r)$ and the cyclic sequence $CS(r)$ of slope r both of which arise from the single relator u_r of the upper presentation $G(K(r)) = \langle a, b \,|\, u_r \rangle$, and review several important properties of these sequences from [4] so that we can adopt small cancellation theory in Section 4.

We first fix some definitions and notation. Let X be a set. By a *word* in X, we mean a finite sequence $x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$ where $x_i \in X$ and $\epsilon_i = \pm 1$. Here we call $x_i^{\epsilon_i}$ the *i-th letter* of the word. For two words u, v in X, by $u \equiv v$ we denote the *visual equality* of u and v, meaning that if $u = x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}$ and $v = y_1^{\delta_1} \cdots y_m^{\delta_m}$ ($x_i, y_j \in X$; $\epsilon_i, \delta_j = \pm 1$), then $n = m$ and $x_i = y_i$ and $\epsilon_i = \delta_i$ for each $i = 1, \dots, n$. For example, two words $x_1 x_2 x_2^{-1} x_3$ and $x_1 x_3$ ($x_i \in X$) are *not* visually equal, though $x_1 x_2 x_2^{-1} x_3$ and $x_1 x_3$ are equal as elements of the free group with basis X. The length of a word v is denoted by $|v|$. A word v in X is said to be *reduced* if v does not contain xx^{-1} or $x^{-1}x$ for any $x \in X$. A word is said to be *cyclically reduced* if all its cyclic permutations are reduced. A *cyclic word* is defined to be the set of all cyclic permutations of a cyclically reduced word. By (v) we denote the cyclic word associated with a cyclically reduced word v. Also by $(u) \equiv (v)$ we mean the *visual equality* of two cyclic words (u) and (v). In fact, $(u) \equiv (v)$ if and only if v is visually a cyclic shift of u.

Definition 3.3. (1) Let v be a reduced word in $\{a, b\}$. Decompose v into

$$v \equiv v_1 v_2 \cdots v_t,$$

where, for each $i = 1, \dots, t-1$, all letters in v_i have positive (resp. negative) exponents, and all letters in v_{i+1} have negative (resp. positive) exponents. Then the sequence of positive integers $S(v) := (|v_1|, |v_2|, \dots, |v_t|)$ is called the *S-sequence of* v.

(2) Let (v) be a cyclic word in $\{a, b\}$. Decompose (v) into

$$(v) \equiv (v_1 v_2 \cdots v_t),$$

where all letters in v_i have positive (resp. negative) exponents, and all letters in v_{i+1} have negative (resp. positive) exponents (taking subindices modulo t). Then the *cyclic* sequence of positive integers $CS(v) := (\!(|v_1|, |v_2|, \dots, |v_t|)\!)$ is called the *cyclic S-sequence of* (v). Here the double parentheses denote that the sequence is considered modulo cyclic permutations.

(3) A reduced word v in $\{a, b\}$ is said to be *alternating* if $a^{\pm 1}$ and $b^{\pm 1}$ appear in v alternately, i.e., neither $a^{\pm 2}$ nor $b^{\pm 2}$ appears in v. A cyclic word (v) is said

to be *alternating* if all cyclic permutations of v are alternating. In the latter case, we also say that v is *cyclically alternating.*

Definition 3.4. For a rational number r with $0 < r \leq 1$, let $G(K(r)) = \langle a, b \,|\, u_r \rangle$ be the upper presentation. Then the symbol $S(r)$ (resp. $CS(r)$) denotes the S-sequence $S(u_r)$ of u_r (resp. cyclic S-sequence $CS(u_r)$ of (u_r)), which is called the *S-sequence of slope* r (resp. the *cyclic S-sequence of slope* r).

Lemma 3.5 ([4, Proposition 4.1 and Lemma 5.2])**.** (1) *An alternating word in $\{a, b\}$ is completely determined by the initial letter and the associated S-sequence.*

(2) *Suppose that r is a rational number with $0 < r \leq 1$. Let w be an arbitrary cyclic permutation of the single relator u_r of the upper presentation of $G(K(r))$. Then the set*

$$\{\textit{the initial letter of } w' \,|\, (w') \equiv (u_r^{\pm 1}) \textit{ and } S(w') = S(w)\}$$

equals $\{a, a^{-1}, b, b^{-1}\}$.

The above lemma implies that the cyclic word $(u_r^{\pm 1})$ is completely determined by the cyclic S-sequence $CS(r)$, to be precise, an alternating word w' is a cyclic permutation of $u_r^{\pm 1}$ if and only if w' satisfies $S(w') = S(w)$ for some w a cyclic permutation of u_r.

Lemma 3.6 ([4, Lemma 4.8 and Proposition 4.2])**.** *Suppose that $r = q/p$ is a rational number with $0 < r \leq 1$, where p and q are relatively prime positive integers. Then the following hold.*

(1) *The sequence $S(r)$ has length $2q$, and its j-th term $s_j(r)$ is given by the following formula* ($1 \leq j \leq 2q$):
$$s_j(r) = \lfloor jp/q \rfloor_* - \lfloor (j-1)p/q \rfloor_*,$$
where $\lfloor x \rfloor_*$ is the greatest integer smaller than x.

(2) *The sequence $S(r)$ represents the cyclic sequence $CS(r)$. Moreover the cyclic sequence $CS(r)$ is invariant by the half-rotation; that is, if $s_j(r)$ denotes the j-th term of $S(r)$ ($1 \leq j \leq 2q$), then $s_j(r) = s_{q+j}(r)$ for every integer j ($1 \leq j \leq q$).*

Remark 3.7. For $r = 0$, we have $S(u_0) = (2)$ by Remark 3.2.

Proposition 3.8 ([4, Propositions 4.3 and 4.5])**.** *Suppose that r is a rational number with $0 < r \leq 1$. Write $r = [m_1, m_2, \dots, m_k]$, where $k \geq 1$, $(m_1, \dots, m_k) \in (\mathbb{Z}_+)^k$ and $m_k \geq 2$ unless $k = 1$. Put $m = m_1$. Then we have the following.*

(1) *Suppose* $k = 1$, *i.e.*, $r = 1/m$. *Then* $S(r) = (m, m)$.
(2) *Suppose* $k \geq 2$. *Then each term of* $S(r)$ *is either* m *or* $m+1$, *and* $S(r)$ *begins with* $m+1$ *and ends with* m.

Moreover, the sequence $S(r)$ *has a decomposition* (S_1, S_2, S_1, S_2) *which satisfies the following.*

(1) *Each* S_i *is symmetric, i.e., the sequence obtained from* S_i *by reversing the order is equal to* S_i. (*Here,* S_1 *is empty if* $k = 1$.)
(2) *Each* S_i *occurs only twice in the cyclic sequence* $CS(r)$.
(3) S_1 *begins and ends with* $m+1$.
(4) S_2 *begins and ends with* m.

Corollary 3.9 ([4, Corollary 4.6])**.** *For a rational number* r *with* $0 < r \leq 1$, $CS(r)$ *is symmetric, i.e., the cyclic sequence obtained from* $CS(r)$ *by reversing its cyclic order is equivalent to* $CS(r)$ (*as a cyclic sequence*).

Lemma 3.10. *Suppose that* r *is a rational number with* $0 < r < 1$. *Let* $S(r) = (S_1, S_2, S_1, S_2)$ *be as in Proposition 3.8. Then, for any* $0 \neq s \in I_1(r) \cup I_2(r)$, *the following hold.*

(1) *The cyclic* S*-sequence* $CS(s)$ *does not contain* (S_1, S_2) *nor* (S_2, S_1) *as a subsequence.*
(2) *If* $r = 1/p$ $(p \geq 2)$, *then* $CS(s)$ *consists of integers less than* p.

In the above lemma (and throughout this paper), we mean by a *subsequence* a subsequence without leap. Namely a sequence $(a_1, a_2, \dots, a_p)$ is called a *subsequence* of a cyclic sequence, if there is a sequence $(b_1, b_2, \dots, b_n)$ representing the cyclic sequence such that $p \leq n$ and $a_i = b_i$ for $1 \leq i \leq p$.

Proof. The first assertion follows from [4, Lemma 7.3 and Remark 5]. To show the second assertion, assume $r = 1/p$. Since $0 \neq s \in I_1(1/p) \cup I_2(1/p)$, s has a continued fraction expansion $s = [l_1, \dots, l_t]$, where $t \geq 1$, $(l_1, \dots, l_t) \in (\mathbb{Z}_+)^t$ and $l_t \geq 2$ unless $t = 1$, such that either both $l_1 = p-1$ and $t = 1$ or both $0 < l_1 \leq p-2$ and $t \geq 1$. If $l_1 = p-1$ and $t = 1$, then $s = 1/(p-1)$ and $CS(s) = ((p-1, p-1))$; so the assertion holds. If $0 < l_1 \leq p-2$ and $t \geq 1$, then each component of $CS(s)$ is equal to $l_1 \leq p-2$ or $l_1 + 1 \leq p-1$ by Proposition 3.8; so the assertion holds. □

Remark 3.11. If $r \neq 1/2$, then the conclusions of Lemma 3.10 holds even for $s = 0$. In fact, if $r \neq 1/2$, then either (i) $r = 1/p$ with $p \geq 3$ and so $S(r) = (p, p) = (S_2, S_2)$, where S_1 is empty, or (ii) both S_1 and S_2 are non-empty. Since $CS(u_0) = ((2))$ by Remark 3.7, this implies the conclusions of

Lemma 3.10 for $s = 0$. However, if $r = 1/2$, then none of the assertions of Lemma 3.10 holds for $s = 0$.

In the remainder of this section, we recall the small cancellation conditions for the 2-bridge link groups established in [4].

Let $F(X)$ be the free group with basis X. A subset R of $F(X)$ is said to be *symmetrized*, if all elements of R are cyclically reduced and, for each $w \in R$, all cyclic permutations of w and w^{-1} also belong to R.

Definition 3.12. Suppose that R is a symmetrized subset of $F(X)$. A nonempty word b is called a *piece* if there exist distinct $w_1, w_2 \in R$ such that $w_1 \equiv bc_1$ and $w_2 \equiv bc_2$. The small cancellation conditions $C(p)$ and $T(q)$, where p and q are integers such that $p \geq 2$ and $q \geq 3$, are defined as follows (see [6]).

(1) Condition $C(p)$: If $w \in R$ is a product of n pieces, then $n \geq p$.
(2) Condition $T(q)$: For $w_1, \dots, w_n \in R$ with no successive elements w_i, w_{i+1} an inverse pair (i mod n), if $n < q$, then at least one of the products $w_1w_2, \dots, w_{n-1}w_n, w_nw_1$ is freely reduced without cancellation.

The following proposition enables us to apply the small cancellation theory to our problem.

Proposition 3.13 ([4, Theorem 5.1])**.** *Suppose that r is a rational number with $0 < r < 1$. Let R be the symmetrized subset of $F(a, b)$ generated by the single relator u_r of the upper presentation of $G(K(r))$. Then R satisfies $C(4)$ and $T(4)$.*

At the end of this section, we recall a key fact concerning the cyclic word (u_r), which is used in the proof of the main theorem in Section 6. (In fact, it is also used in the proof of Proposition 3.13 above and that of Lemma 4.10 implicitly.)

Definition 3.14. For a positive integer n, a non-empty subword w of the cyclic word (u_r) is called a *maximal n-piece* if w is a product of n pieces and if any subword w' of u_r which properly contains w as an *initial* subword is not a product of n-pieces.

Lemma 3.15 ([4, Corollary 5.4(1)])**.** *Let $p \geq 2$ be an integer, and let $u_{1/p}$ be the single relator of the upper presentation of $G(K(1/p))$. Recall that $S(1/p) = S(u_{1/p}) = (p, p)$, and decompose*

$$u_{1/p} \equiv v_2v_4,$$

where $S(v_2) = S(v_4) = (p)$. Let v_{ib}^* be the maximal proper initial subword of v_i, i.e., the initial subword of v_i such that $|v_{ib}^*| = |v_i| - 1$ $(i = 2, 4)$. Then the following hold, where v_{ib} and v_{ie} are nonempty initial and terminal subwords of v_i with $|v_{ib}|, |v_{ie}| \leq |v_i| - 1$, respectively.

(1) *The following is the list of all maximal 1-pieces of $(u_{1/p})$, arranged in the order of the position of the initial letter:*

$$v_{2b}^*, v_{2e}, v_{4b}^*, v_{4e}.$$

(2) *The following is the list of all maximal 2-pieces of $(u_{1/p})$, arranged in the order of the position of the initial letter:*

$$v_2, v_{2e}v_{4b}^*, v_4, v_{4e}v_{2b}^*.$$

4. Annular diagrams over 2-bridge link groups

In this section, we establish a very strong structure theorem (Theorem 4.9 and Corollary 4.11) for the annular diagram which arises in the study of the conjugacy problem by using the small cancellation theory. The structure theorem forms a foundation of the whole papers in this series.

Let us begin with necessary definitions and notation following [6]. A *map* M is a finite 2-dimensional cell complex embedded in $\mathbb{R}^2$. To be precise, M is a finite collection of vertices (0-cells), edges (1-cells), and faces (2-cells) in $\mathbb{R}^2$ satisfying the following conditions.

(i) A vertex is a point in $\mathbb{R}^2$.
(ii) An edge is homeomorphic to an open interval such that $\bar{e} = e \cup \{a\} \cup \{b\}$, where a and b are vertices of M which are possibly identical.
(iii) For each face D of M, there is a continuous map f from the 2-ball B^2 to $\mathbb{R}^2$ such that
  (a) the restriction of f to the interior of B^2 is a homeomorphism onto D, and
  (b) the image of ∂B^2 is equal to $\cup_{i=1}^n \bar{e}_i$ for some set $\{e_1, \dots, e_n\}$ of edges of M.

The underlying space of M, i.e., the union of the cells in M, is also denoted by the same symbol M. The boundary (frontier), ∂M, of M in $\mathbb{R}^2$ is regarded as a 1-dimensional subcomplex of M. An edge may be traversed in either of two directions. If v is a vertex of a map M, $d_M(v)$, the *degree of* v, will denote the number of oriented edges in M having v as initial vertex.

A *path* in M is a sequence of oriented edges $e_1, \dots, e_n$ such that the initial vertex of e_{i+1} is the terminal vertex of e_i for every $1 \leq i \leq n-1$. A *cycle* is a closed path, namely a path $e_1, \dots, e_n$ such that the initial vertex of e_1 is the

terminal vertex of e_n. If D is a face of M, any cycle of minimal length which includes all the edges of the boundary, ∂D, of D is called a *boundary cycle* of D. By $d_M(D)$, the *degree of* D, we denote the number of oriented edges in a boundary cycle of D.

Definition 4.1. A non-empty map M is called a $[p,q]$*-map* if the following conditions hold.

(i) $d_M(v) \geq p$ for every vertex $v \in M - \partial M$.
(ii) $d_M(D) \geq q$ for every face $D \in M$.

Definition 4.2. Let R be a symmetrized subset of $F(X)$. An R*-diagram* is a map M and a function ϕ assigning to each oriented edge e of M, as a *label*, a reduced word $\phi(e)$ in X such that the following hold.

(i) If e is an oriented edge of M and e^{-1} is the oppositely oriented edge, then $\phi(e^{-1}) = \phi(e)^{-1}$.
(ii) For any boundary cycle δ of any face of M, $\phi(\delta)$ is a cyclically reduced word representing an element of R. (If $\alpha = e_1, \dots, e_n$ is a path in M, we define $\phi(\alpha) \equiv \phi(e_1) \cdots \phi(e_n)$.)

Let D_1 and D_2 be faces (not necessarily distinct) of M with an edge $e \subseteq \partial D_1 \cap \partial D_2$. Let $e\delta_1$ and $\delta_2 e^{-1}$ be boundary cycles of D_1 and D_2, respectively. Let $\phi(\delta_1) = f_1$ and $\phi(\delta_2) = f_2$. An R-diagram M is said to be *reduced* if one never has $f_2 = f_1^{-1}$.

It is easy to observe that we may assume the following convention (see [4, Convention 1]).

Convention 4.3. Let R be the symmetrized subset of $F(a,b)$ generated by the single relator u_r of the upper presentation of $G(K(r))$. For any R-diagram M, we assume that M satisfies the following.

(1) $d_M(v) \geq 3$ for every vertex $v \in M - \partial M$.
(2) For every edge e of ∂M, the label $\phi(e)$ is a piece.
(3) For a path $e_1, \dots, e_n$ in ∂M of length $n \geq 2$ such that the vertex $\bar{e}_i \cap \bar{e}_{i+1}$ has degree 2 for $i = 1, 2, \dots, n-1$, $\phi(e_1)\phi(e_2)\cdots\phi(e_n)$ cannot be expressed as a product of less than n pieces.

The following corollary is immediate from Proposition 3.13 and Convention 4.3.

Corollary 4.4 ([4, Corollary 6.2])**.** *Suppose that r is a rational number with $0 < r < 1$. Let R be the symmetrized subset of $F(a,b)$ generated by the single relator u_r of the upper presentation of $G(K(r))$. Then every reduced R-diagram is a $[4,4]$-map.*

We turn to interpreting conjugacy in terms of diagrams.

Definition 4.5. An *annular map* M is a connected map such that $\mathbb{R}^2 - M$ has exactly two connected components. For a symmetrized subset R of $F(a,b)$, an *annular R-diagram* is an R-diagram whose underlying map is an annular map.

Let M be an annular R-diagram, and let K and H be, respectively, the unbounded and bounded components of $\mathbb{R}^2 - M$. We call $\partial K (\subset \partial M)$ the *outer boundary* of M, while $\partial H (\subset \partial M)$ is called the *inner boundary* of M. Clearly, the *boundary* of M, ∂M, is the union of the outer boundary and the inner boundary. A cycle of minimal length which contains all the edges in the outer (inner, resp.) boundary of M going around once along the boundary of K (H, resp.) is an *outer (inner, resp.) boundary cycle* of M. An *outer (inner, resp.) boundary label of* M is defined to be a word $\phi(\alpha)$ in X for α an outer (inner, resp.) boundary cycle of M.

Convention 4.6. Since M is embedded in $\mathbb{R}^2$, each 2-cell of M inherits an orientation of $\mathbb{R}^2$. Throughout this series of papers, we assume, unlike the usual orientation convention, that $\mathbb{R}^2$ is oriented so that the boundary cycles of the 2-cells of M are clockwise. Thus the outer boundary cycles are clockwise and inner boundary cycles are counterclockwise, unlike the convention in [6, p.253].

The following lemma is a well-known classical result in combinatorial group theory.

Lemma 4.7 ([6, Lemmas V.5.1 and V.5.2])**.** *Suppose $G = \langle X \,|\, R \rangle$ with R being symmetrized. Let u, v be two cyclically reduced words in X which are not trivial in G and which are not conjugate in $F(X)$. Then u and v represent conjugate elements in G if and only if there exists a reduced annular R-diagram M such that u is an outer boundary label and v^{-1} is an inner boundary label of M.*

The following lemma will play an essential role in the proof of Theorem 4.9 below. This lemma is easily obtained from the arguments of [6, Theorem V.3.1].

Lemma 4.8 (cf. [6, Theorem V.3.1])**.** *Let M be an arbitrary connected map. Then*

$$4 - 4h \leq \sum_{v \in \partial M} (3 - d_M(v)) + \sum_{v \in M - \partial M} (4 - d_M(v)) + \sum_{D \in M} (4 - d_M(D)),$$

where h is the number of holes of M, i.e., the number of bounded components of $\mathbb{R}^2 - M$. In particular, if M is a $[4,4]$-map, then

$$4 - 4h \leq \sum_{v \in \partial M} (3 - d_M(v)).$$

Proof. We repeat the proof of a part of [6, Theorem V.3.1]. Put

$V =$ the number of vertices of M;

$E =$ the number of (unoriented) edges of M;

$F =$ the number of faces of M;

$V^{\bullet} =$ the number of vertices in ∂M;

$E^{\bullet} =$ the number of (unoriented) edges in ∂M, *counted with multiplicity.*

To be more precise, $E^{\bullet}$ means the number of (unoriented) edges on ∂M with an edge counted twice if it appears twice in the cycles necessary to describe the boundary of M. For example, the edge e_0 in each of Figure 3(a) and Figure 3(b) has to be counted twice in computing $E^{\bullet}$, because e_0 in Figure 3(a) occurs twice in a boundary cycle of M and e_0 in Figure 3(b) occurs in both inner and outer boundary cycles of M. However the vertex v_0 in each of Figure 3(a) and Figure 3(b) has to be counted only once in computing $V^{\bullet}$.

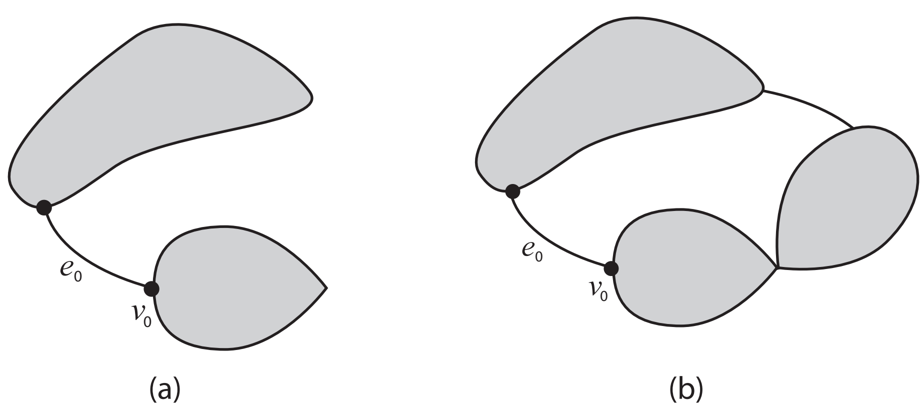


FIGURE 3. The edge e_0 is counted twice in computing $E^{\bullet}$, while the vertex v_0 is counted once in computing $V^{\bullet}$.

Since M has h holes, it follows from Euler's Formula that

$$1 - h = V - E + F. \tag{†}$$

Putting $d(v) = d_M(v)$ for vertices v of M and $d(D) = d_M(D)$ for faces D of M, it is easy to observe that

$$2E = \sum_{v\in M} d(v);$$
$$2E = \sum_{D\in M} d(D) + E^{\bullet}.$$

These equations together with (†) yield

$$\begin{aligned}
4-4h &= 4V + 4F - \sum_{v\in M} d(v) - \sum_{D\in M} d(D) - E^{\bullet}\\
&= \sum_{v\in M}(4-d(v)) + \sum_{D\in M}(4-d(D)) - E^{\bullet}\\
&= \sum_{v\in\partial M}(4-d(v)) + \sum_{v\in M-\partial M}(4-d(v)) + \sum_{D\in M}(4-d(D)) - E^{\bullet}\\
&= \sum_{v\in\partial M}(3-d(v)) + \sum_{v\in M-\partial M}(4-d(v)) + \sum_{D\in M}(4-d(D)) + V^{\bullet} - E^{\bullet}.
\end{aligned}$$

Note that

$$V^{\bullet} - E^{\bullet} \le V^{\bullet} - E_0^{\bullet} = \chi(\partial M) = \beta_0(\partial M) - \beta_1(\partial M) \le 0,$$

where $E_0^{\bullet}$ is the number of (unoriented) edges in ∂M (counted *without* multiplicity), χ denotes the Euler characteristic, and β_i denotes the i-th Betti number. The last inequality follows from the fact that each component C of ∂M has a positive first Betti number, which in turn follows from the fact that $\mathbb{R}^2 - C$ is not connected. Hence,

$$4-4h \le \sum_{v\in\partial M}(3-d(v)) + \sum_{v\in M-\partial M}(4-d(v)) + \sum_{D\in M}(4-d(D)),$$

as required. The remaining assertion follows from the above inequality and the fact that if M is a $[4,4]$-map then

$$\sum_{v\in M-\partial M}(4-d(v)) \le 0 \quad\text{and}\quad \sum_{D\in M}(4-d(D)) \le 0.$$

□

For a rational number r with $0 < r < 1$, let R be the symmetrized subset of $F(a,b)$ generated by the single relator u_r of the upper presentation of $G(K(r))$. Our goal of the present section is to describe the structure of reduced annular R-diagrams with u_s and $u_{s'}^{\pm 1}$, respectively, as outer and inner boundary labels,

where u_s and $u_{s'}$ $(s, s' \in I_1(r) \cup I_2(r))$ are the cyclically reduced words in $\{a, b\}$ obtained from the simple loops α_s and $\alpha_{s'}$, respectively, as in Lemma 3.1. We obtain the following very strong structure theorem.

Theorem 4.9 (Structure Theorem)**.** *Suppose that r is a rational number with $0 < r < 1$. Let R be the symmetrized subset of $F(a, b)$ generated by the single relator u_r of the upper presentation of $G(K(r))$, and let $S(r) = (S_1, S_2, S_1, S_2)$ be as in Proposition 3.8. Suppose that M is a reduced annular R-diagram such that*

(i) *the words $\phi(\alpha)$ and $\phi(\delta)$ are cyclically reduced;*
(ii) *the words $\phi(\alpha)$ and $\phi(\delta)$ are cyclically alternating;*
(iii) *the cyclic S-sequences of the cyclic words $(\phi(\alpha))$ and $(\phi(\delta))$ do not contain (S_1, S_2) nor (S_2, S_1) as a subsequence,*

where α and δ are, respectively, arbitrary outer and inner boundary cycles of M. Let the outer and inner boundaries of M be denoted by σ and τ, respectively. Then the following hold.

(1) *The outer and inner boundaries σ and τ are simple, i.e., they are homeomorphic to the circle, and there is no edge contained in $\sigma \cap \tau$.*
(2) *$d_M(v) = 2$ or 4 for every vertex $v \in \partial M$. Moreover, on both σ and τ, vertices of degree 2 appear alternately with vertices of degree 4.*
(3) *$d_M(v) = 4$ for every vertex $v \in M - \partial M$.*
(4) *$d_M(D) = 4$ for every face $D \in M$.*

Before proving the theorem, we prepare the following lemma.

Lemma 4.10. *Under the assumption of Theorem 4.9, the following hold.*

(1) *There is no face D in M such that $\partial D \cap \partial M$ contains three edges e_1, e_2 and e_3 such that $\bar{e}_1 \cap \bar{e}_2 = \{v_1\}$ and $\bar{e}_2 \cap \bar{e}_3 = \{v_2\}$, where $d_M(v_i) = 2$ for each $i = 1, 2$.*
(2) *The outer and inner boundaries σ and τ are simple.*

Proof. (1) Suppose on the contrary that there is a face D in M and edges e_1, e_2 and e_3 in $\partial D \cap \partial M$ satisfying the condition. Then e_1, e_2 and e_3 form three consecutive edges in the outer or inner boundary cycle, say the outer boundary cycle α. On the other hand, we can see that $\phi(e_1)\phi(e_2)\phi(e_3)$ contains a subword w such that the S-sequence of w is (S_1, S_2, ℓ) or (ℓ, S_2, S_1), for some positive integer ℓ, as follows. Let w_0 be the maximal 2-piece which forms a proper initial subword of $\phi(e_1)\phi(e_2)\phi(e_3)$. By using the classification of the maximal 2-pieces in [4, Corollary 5.4] (cf. Lemma 3.15), we can confirm the condition for all possibilities for w_0 (see the argument in the ending of the

proof of [4, Theorem 6.3]). By the proof of [4, Corollary 6.4], this implies that the cyclic S-sequence of $(\phi(\alpha))$ contains (S_1, S_2) or (S_2, S_1). This contradicts the hypothesis (iii).

(2) Suppose on the contrary that σ or τ is not simple. Then there is an extremal disk, say J, which is properly contained in M and connected to the rest of M by a single vertex. Here, recall that an *extremal disk* of a map M is a submap of M which is topologically a disk and which has a boundary cycle $e_1, \dots, e_n$ such that the edges $e_1, \dots, e_n$ occur in order in some boundary cycle of the whole map M. But then, by Claim in the proof of [4, Theorem 6.3], there is a face D in M such that $\partial D \cap \partial M$ contains three consecutive edges e_1, e_2 and e_3 such that $\bar{e}_1 \cap \bar{e}_2 = \{v_1\}$ and $\bar{e}_2 \cap \bar{e}_3 = \{v_2\}$, where $d_M(v_i) = d_J(v_i) = 2$ for each $i = 1, 2$, contradicting Lemma 4.10(1). $\square$

Proof of Theorem 4.9. Note first that M is a connected annular $[4, 4]$-map by Corollary 4.4. By hypothesis (i), there is no vertex of degree 1 in ∂M. Moreover, there is no vertex of degree 3 in ∂M, as is shown in the following. Suppose there is a vertex $v \in \partial M$ of degree 3. Then there are at most two faces which contain v. If there are two such faces, then one of $\phi(\alpha)$ and $\phi(\delta)$ is not cyclically alternating, a contradiction to hypothesis (ii). If there is a unique such face, then by using hypothesis (ii), we see that the boundary label of the face is not cyclically alternating, a contradiction. Finally, if there is no such face, then this contradicts hypothesis (ii). Hence, no vertex in ∂M has degree 1 nor 3, and so every vertex in ∂M must have degree 2 or at least 4.

We argue two cases separately.

Case 1. $\sigma \cap \tau = \emptyset$.

In this case, (1) follows immediately from Lemma 4.10(2).

(2) Since ∂M is the disjoint union of σ and τ, Lemma 4.8 yields

$$0 \leq \sum_{v \in \sigma} (3 - d_M(v)) + \sum_{v \in \tau} (3 - d_M(v)).$$

On the other hand, since σ and τ are simple by Lemma 4.10(2) and since they are disjoint by the current assumption, every edge in σ or τ is contained in the boundary of a unique face of M. Thus Lemma 4.10(1) implies that vertices of degree 2 do not occur consecutively on σ nor on τ. Hence, the above inequality holds only when $d_M(v) = 2$ or $d_M(v) = 4$ for every vertex $v \in \sigma \cup \tau$ and when vertices of degree 2 appear alternately with vertices of degree 4 on both σ and τ, thus proving (2).

(3)–(4) By (2), $\sum_{v \in \partial M}(3 - d_M(v)) = 0$. This together with Lemma 4.8 yields

$$0 \leq \sum_{v \in M - \partial M} (4 - d_M(v)) + \sum_{D \in M} (4 - d_M(D)).$$

Here, since $4 - d_M(v) \leq 0$ for every $v \in M - \partial M$ and $4 - d_M(D) \leq 0$ for every $D \in M$ by the definition of a $[4,4]$-map, the only possibility is that $4 - d_M(v) = 0$ for every vertex $v \in M - \partial M$ and $4 - d_M(D) = 0$ for every face $D \in M$, thus proving (3) and (4).

Case 2. $\sigma \cap \tau \neq \emptyset$.

(1) Suppose on the contrary that $\sigma \cap \tau$ contains an edge. As illustrated in Figure 4, there is a submap J of M such that

- (i) J is bounded by a simple closed path of the form $\sigma_1\tau_1$, where $\sigma_1 \subseteq \sigma$ and $\tau_1 \subseteq \tau$;
- (ii) J is connected to the rest of M by two distinct vertices, say v_1 and v_2, where $\sigma_1 \cap \tau_1 = \{v_1, v_2\}$ and v_1 is an endpoint of an edge contained in $\sigma \cap \tau$. Note that $d_J(v_1) = d_M(v_1) - 1 \geq 3$ and $d_J(v_2) \geq 2$.

Since J is a connected and simply connected $[4,4]$-map, Lemma 4.8 yields

$$4 \leq \sum_{v \in \partial J} (3 - d_J(v)).$$

Since $d_J(v_1) \geq 3$ and $d_J(v_2) \geq 2$, this inequality implies

$$3 \leq \sum_{v \in \partial J - \{v_1, v_2\}} (3 - d_J(v)).$$

On the other hand, since every vertex in $\partial J - \{v_1, v_2\}$ has degree 2 or at least 4 and since degree 2 vertices cannot occur consecutively on $\sigma_1 - \{v_1, v_2\}$ nor on $\tau_1 - \{v_1, v_2\}$ (see Lemma 4.10(1)),

$$\sum_{v \in \partial J - \{v_1, v_2\}} (3 - d_J(v)) \leq 2,$$

a contradiction.

(2)–(4) By (1), $\sigma \cap \tau$ consists of finitely many vertices in M. Then, by Lemma 4.10(1), vertices of degree 2 do not occur consecutively on σ nor on τ.

First suppose that $\sigma \cap \tau$ consists of a single vertex, say v_0. Cut M open at v_0 to get a connected and simply connected $[4,4]$-map M'. In this process, the vertex v_0 is separated into two distinct vertices, say v_0' and v_0'', in M' such that $d_{M'}(v_0'), d_{M'}(v_0'') \geq 2$ and $d_{M'}(v_0') + d_{M'}(v_0'') = d_M(v_0)$. Then M' is bounded

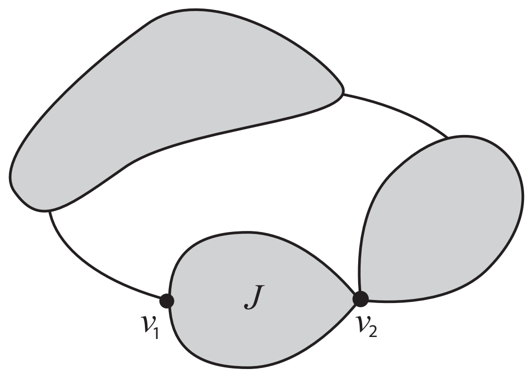


FIGURE 4. A possible annular map M when $\sigma \cap \tau$ contains an edge.

by a simple closed path of the form $\sigma_0\tau_0$, where $\sigma_0 \cap \tau_0 = \{v_0', v_0''\}$. Again by Lemma 4.8,

$$4 \leq \sum_{v \in \partial M'} (3 - d_{M'}(v)).$$

Considering that vertices of degree 2 do not occur consecutively on $\sigma_0 - \{v_0', v_0''\}$ nor on $\tau_0 - \{v_0', v_0''\}$ (see Lemma 4.10(1)), we see that only the equality can hold, and that the equality holds only when $d_{M'}(v) = 2$ or 4 for every vertex $v \in \sigma_0 \cup \tau_0 - \{v_0', v_0''\}$, $d_{M'}(v_0') = 2 = d_{M'}(v_0'')$ and when vertices of degree 2 appear alternately with vertices of degree 4 starting and ending with vertices of degree 2 on both $\sigma_0 - \{v_0', v_0''\}$ and $\tau_0 - \{v_0', v_0''\}$. This implies that $d_M(v) = 2$ or 4 for every vertex $v \in \partial M - \{v_0\}$, $d_M(v_0) = 4$, and that vertices of degree 2 appear alternately with vertices of degree 4 on both σ and τ, thus proving (2).

Since $\sum_{v \in \partial M'}(3 - d_{M'}(v)) = 4$, Lemma 4.8 yields

$$0 \leq \sum_{v \in M' - \partial M'} (4 - d_{M'}(v)) + \sum_{D \in M'} (4 - d_{M'}(D)).$$

Here, by the definition of a $[4,4]$-map, $4 - d_{M'}(v) \leq 0$ for every vertex $v \in M' - \partial M'$ and $4 - d_{M'}(D) \leq 0$ for every face $D \in M'$, we must have $d_M(v) = d_{M'}(v) = 4$ for every vertex $v \in M' - \partial M'$ and $d_M(D) = d_{M'}(D) = 4$ for every face $D \in M'$, thus proving (3) and (4).

Next suppose that $\sigma \cap \tau$ consists of at least two vertices, say $v_1, \ldots, v_n$, where $n \geq 2$ and where these vertices are indexed according as there is a submap J_i of M for every $i = 1, \ldots, n$ such that

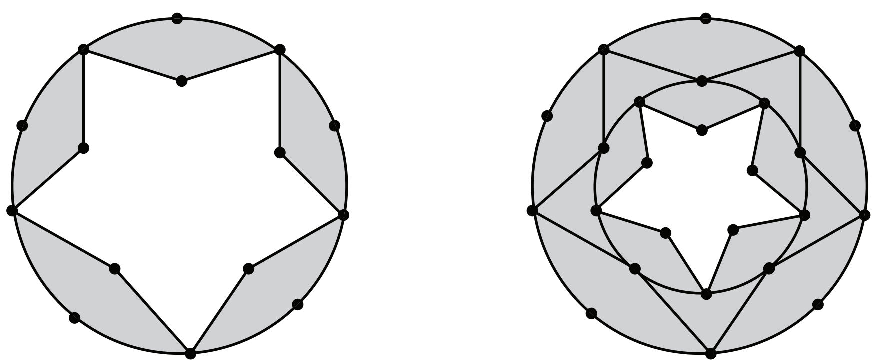
(i) J_i is bounded by a simple closed path of the form $\sigma_i\tau_i$, where $\sigma_i \subseteq \sigma$ and $\tau_i \subseteq \tau$;

(ii) J_i is connected to the rest of M by two distinct vertices, say v_i and v_{i+1}, where $\sigma_i \cap \tau_i = \{v_i, v_{i+1}\}$ and where $d_{J_i}(v_i), d_{J_i}(v_{i+1}) \geq 2$ and $d_{J_i}(v_{i+1}) + d_{J_{i+1}}(v_{i+1}) = d_M(v_{i+1})$ (taking the indices modulo n).

Then each J_i is a connected and simply connected $[4,4]$-map such that $M = J_1 \cup \cdots \cup J_n$. Moreover $\sigma = \sigma_1 \cup \cdots \cup \sigma_n$ and $\tau = \tau_1 \cup \cdots \cup \tau_n$. The same arguments as for (M', v_0', v_0'') above applies to each (J_i, v_i, v_{i+1}) to prove the assertions. □

Theorem 4.9 enables us to identify all possible shapes of the annular maps M. To describe the result, we define the *outer boundary layer* of an annular map M to be a submap of M consisting of all faces D such that the intersection of ∂D with the outer boundary of M contains an edge, together with the edges and vertices contained in ∂D.

Corollary 4.11. *Let M be a reduced annular $[4,4]$-map satisfying the assumptions of Theorem 4.9. Then Figure 5(a) illustrates the only possible type of the outer boundary layer of M, while Figure 5(b) illustrates the only possible type of whole M. (The number of faces per layer and the number of layers are variable.)*



(a) (b)

Figure 5.

Proof. Let $v_1, v_2, \ldots, v_{2n}$ be the vertices of the outer boundary σ arranged in this cyclic order in σ, such that $d_M(v_i)$ is 4 or 2 according as i is odd or even (see Theorem 4.9(1) and (2)). Let e_i $(1 \leq i \leq 2n)$ be the oriented edge in σ

running from v_i to v_{i+1}, where the indices are considered modulo $2n$. Then, for each j ($1 \le j \le n$), there is a unique face, D_j, of M whose boundary contains the vertices v_{2j-1}, v_{2j} and v_{2j+1} and the edges e_{2j-1} and e_{2j}. Since $d_M(D_j) = 4$ by Theorem 4.9(4), there is a unique vertex in ∂D_j different from v_{2j-1}, v_{2j} and v_{2j+1}. We denote it by v'_{2j}, and let e'_{2j-1} (resp. e'_{2j}) be the oriented edges in ∂D_j running from v_{2j-1} to v'_{2j} (resp. from v'_{2j} to v_{2j+1}). Then the cycle $e_{2j-1}, e_{2j}, e'^{-1}_{2j}, e'^{-1}_{2j-1}$ is a boundary cycle of D_j.

Claim 1. *The unoriented edges determined by e'_i and e'_{i+1} are distinct for every $1 \le i \le 2n$, where indices are taken modulo $2n$.*

Proof of Claim 1. Suppose this is not the case. Then we have $e'_i = e'^{-1}_{i+1}$ for some i. Suppose first that $e'_{2j} = e'^{-1}_{2j+1}$ for some j. Then the union of the closures $\bar{D}_j$ and $\bar{D}_{j+1}$ forms a neighborhood of v_{2j+1} in M and hence, e_{2j}, e^{-1}_{2j+1} and $e'_{2j} = e'^{-1}_{2j+1}$ are the only edges of M having v_{2j+1} as the terminal point. So $d_M(v_{2j+1}) = 3$, a contradiction to Theorem 4.9(2). Next, suppose $e'_{2j-1} = e'^{-1}_{2j}$. Then v'_{2j} is contained in the interior of the closure $\bar{D}_j$, and $e'_{2j-1} = e'^{-1}_{2j}$ is the only edge of M having v'_{2j} as the terminal point. So $d_M(v'_{2j}) = 1$, a contradiction to Theorem 4.9(3). $\square$

Claim 2. *e'_i is not contained in σ for every $1 \le i \le 2n$.*

Proof of Claim 2. We prove the claim when $i = 2j$ for some integer j. (The remaining case can be treated by a parallel argument.) Suppose e'_{2j} is contained in σ. Then, since v_{2j+1} is the terminal point of e'_{2j}, the oriented edge e'_{2j} is equal to e_{2j} or e^{-1}_{2j+1}. Since D_j is a planner 2-cell, $e'_{2j} = e_{2j}$ cannot happen. So we have $e'_{2j} = e^{-1}_{2j+1}$. Then v_{2j+2} is the terminal point of the oriented edges $e_{2j+1} = e'^{-1}_{2j}$, e^{-1}_{2j+2} and e'_{2j-1}. Since $e'_{2j-1} \ne e'^{-1}_{2j}$ by Claim 1 and since $d_M(v_{2j+2}) = 2$, this implies $e'_{2j-1} = e^{-1}_{2j+2}$. In particular, $v_{2j-1} = v_{2j+3}$ and therefore $n = 1$ or 2. By the above observations, we see that the boundary cycle $e_{2j-1}, e_{2j}, e'^{-1}_{2j}, e'^{-1}_{2j-1}$ of D_j is equal to $e_{2j-1}, e_{2j}, e_{2j+1}, e_{2j+2}$. If $n = 1$, then we have $j = 1$ and the boundary cycle of D_1 is equal to e_1, e_2, e_1, e_2. This contradicts the fact that D_1 is a planner 2-cell. If $n = 2$, then the boundary cycle of D_j is equal to e_1, e_2, e_3, e_4 up to cyclic permutation and hence $\partial D_j = \sigma$. This contradicts the assumption that M is an annular diagram. $\square$

Claim 3. *For every $1 \le j \le n$, the oriented edges $e_{2j}, e^{-1}_{2j+1}, e'_{2j}, e'^{-1}_{2j+1}$ are mutually distinct, and they are the only oriented edges of M which have v_{2j+1} as the terminal point.*

Proof of Claim 3. This follows from Claims 1 and 2 and the fact that $d_M(v_{2j+1}) = 4$. $\square$

Claim 4. *$D_j \neq D_h$ for every $1 \leq j \neq h \leq n$.*

Proof of Claim 4. Suppose $D_j = D_h$ for some $1 \leq j \neq h \leq n$. By Claim 2, e_{2j-1} and e_{2j} are the only edges contained in $\sigma \cap \partial D_j$, whereas e_{2h-1} and e_{2h} are the only edges contained in $\sigma \cap \partial D_h$. Thus we see e_{2j-1} is equal to e_{2h-1} or e_{2h}. Since σ is simple and since $j \neq h$, this is impossible. $\square$

Claim 5. *v'_{2j} is not contained in σ for every $1 \leq j \leq n$.*

Proof of Claim 5. Suppose v'_{2j} is contained in σ for some j. Then $v'_{2j} = v_k$ for some $1 \leq k \leq 2n$, and the oriented edges $e_{k-1}, e_k^{-1}, e'_{2j-1}, e'^{-1}_{2j}$ have $v'_{2j} = v_k$ as the terminal point. Since these are mutually distinct by Claims 1 and 2, we have $d_M(v_k) \geq 4$ and hence $k = 2h+1$ for some integer h. If $h = j$, then e'_{2j} is a loop based on $v'_{2j} = v_{2j+1}$ and hence $d_M(v_{2j+1}) \geq 5$, a contradiction. Similarly, we see $h = j-1$ cannot happen. Hence we have $v'_{2j} = v_{2h+1}$, where $h \neq j, j-1$.

Note that in addition to $e_{2h}, e^{-1}_{2h+1}, e'_{2h}, e'^{-1}_{2h+1}$, the oriented edges e'_{2j-1} and e'^{-1}_{2j} have $v'_{2j} = v_{2h+1}$ as the terminal point. By Claims 1 and 2, this implies that the pair $(e'_{2j-1}, e'^{-1}_{2j})$ is equal to $(e'_{2h}, e'^{-1}_{2h+1})$ or $(e'^{-1}_{2h+1}, e'_{2h})$. We first show that $(e'_{2j-1}, e'^{-1}_{2j}) = (e'_{2h}, e'^{-1}_{2h+1})$ cannot happen. To this end, note that the 2-cell D_h (resp. D_j) lies on the left hand side of the oriented edge e'_{2h} (resp. e'_{2j-1}), by virtue of Convention 4.6. So, if $(e'_{2j-1}, e'^{-1}_{2j}) = (e'_{2h}, e'^{-1}_{2h+1})$ holds, then $e'_{2j-1} = e'_{2h}$ and hence we must have $D_j = D_h$. This contradicts Claim 4. So we may assume $(e'_{2j-1}, e'^{-1}_{2j}) = (e'^{-1}_{2h+1}, e'_{2h})$. Then the initial point v_{2j+1} of e'^{-1}_{2j} is equal to the initial point v'_{2h} of e'_{2h}. Similarly, we have $v_{2j-1} = v'_{2h+2}$. Thus we have shown that the identity $v'_{2j} = v_{2h+1}$ implies the identities $v'_{2h} = v_{2j+1}$ and $v'_{2h+2} = v_{2j-1}$. By repeatedly applying this fact, we see that, for every integer k, $v'_{2(h-k)} = v_{2(j+k)+1}$ and $v'_{2(j+k)} = v_{2(h-k)+1}$. Thus we can find a pair of integers j^* and h^* such that $v'_{2j^*} = v_{2h^*+1}$ and $h^* = j^*$ or $j^* - 1$. However, this is impossible by the argument in the first paragraph of this proof. $\square$

Claim 6. *The unoriented edges determined by e'_i $(1 \leq i \leq 2n)$ are mutually distinct.*

Proof of Claim 6. Suppose this is not the case. Then $e'_i = e'_k$ or $e'_i = e'^{-1}_k$ for some $1 \leq i < k \leq 2n$. If $e'_i = e'_k$, then $D_{i'} = D_{k'}$ for some $i' \neq k'$ as in the proof of Claim 5, a contradiction. So we see $e'_i = e'^{-1}_k$. If necessary by inverting the

suffix, we may assume $i = 2j$ for some integer j. By Claim 5, we see $k = 2h+1$ for some integer h, and hence $e'_{2j} = e'^{-1}_{2h+1}$. Thus the terminal point v_{2j+1} of e'_{2j} is equal to the terminal point v_{2h+1} of e'^{-1}_{2h+1}. Since σ is simple, this implies $j = h$ and hence $e'_{2j} = e'^{-1}_{2j+1}$, a contradiction to Claim 1 (or Claim 3). □

Claim 7. *The vertices v'_{2j} $(1 \le j \le n)$ are mutually distinct.*

Proof of Claim 7. Suppose $v'_{2j} = v'_{2h}$ for some $1 \le j \ne h \le n$. Then e'_{2j-1}, e'^{-1}_{2j}, e'_{2h-1}, e'^{-1}_{2h} have $v'_{2j} = v'_{2h}$ as the terminal point. Since these oriented edges are mutually distinct by Claim 6 and since $d_M(v'_{2j}) = 2$ or 4 by Theorem 4.9(2),(3), we see that $d_M(v'_{2j}) = 4$ and that these are the only oriented edges having $v'_{2j} = v'_{2h}$ as the terminal point. We can choose j and h, so that they are *outermost* in the sense that $1 \le j < h \le n$ (after a cyclic permutation of indices) and that the vertices $v'_{2j} = v'_{2h}$ and v'_{2k} $(j < k < h)$ are mutually distinct. Then $\cup_{2j \le i \le 2h-1} \bar{e}'_i$ is a simple loop. We show that e'_{2j}, e'_{2h-1} are contained in a boundary cycle of the annular diagram M. Suppose this is not the case. Then, by the above observations, there is a face, D, of M whose boundary cycle contains e'_{2j}, e'_{2h-1}. By Claim 3, the edges e'_{2j+1} and e'_{2h-2} must be contained in ∂D. Since $d_M(D) = 4$ by Theorem 4.9(4), a boundary cycle of D is given by e'_{2j}, e'_{2j+1}, e'_{2h-2}, e'_{2h-1}. So, $h = j+2$ and v'_{2j+2} is contained in the interior of $\bar{D} \cup \bar{D}_{j+1}$. This implies that v'_{2j+2} is an inner vertex and $d_M(v'_{2j+2}) = 2$, a contradiction to Theorem 4.9(3). Hence e'_{2j}, e'_{2h-1} are contained in a boundary cycle of the annular diagram M, and in particular, $v_{2h-1}, v'_{2h} = v'_{2j}, v_{2j+1}$ lie in the inner boundary of M successively. However all of them have degree 4. This contradicts Theorem 4.9(2) □

By Claims 6 and 7, we see that $\sigma' := \cup_{i=1}^{2n} \bar{e}'_i$ is a simple loop and that $\sigma \cap \sigma' = \{v_{2j+1} \mid 0 \le j \le n-1\}$. Thus the outer boundary layer, J, of M is as illustrated in Figure 5(a). By repeating the argument in the proof of the above claim to the closure of $M - J$, we obtain the desired result. □

5. Proof of the if part of Main Theorem 2.7

Let $p \ge 2$ be an integer and suppose that two distinct elements $s, s' \in I_1(1/p) \cup I_2(1/p)$ satisfy $s = q_1/p_1$ and $s' = q_1/(pq_1 - p_1)$, where (p_1, q_1) is a pair of relatively prime positive integers. Let u_s and $u_{s'}$ be the cyclically reduced words in $\{a, b\}$ obtained from the simple loops α_s and $\alpha_{s'}$, respectively, as in Lemma 3.1. To prove that the unoriented loops α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(1/p)$, it suffices to show that u_s and $u_{s'}^{\pm 1}$ are conjugate in $G(K(1/p))$. By Lemma 3.1, u_s and $u_{s'}$ are cyclically alternating, and both have the initial

letter a. Since $0 \neq s \in I_1(1/p) \cup I_2(1/p)$, s has a continued fraction expansion $s = [l_1, \dots, l_t]$, where $t \geq 1$, $(l_1, \dots, l_t) \in (\mathbb{Z}_+)^t$ and $l_t \geq 2$ unless $t = 1$, such that either both $l_1 = p-1$ and $t = 1$ or both $0 < l_1 \leq p-2$ and $t \geq 1$. We consider two cases separately.

Case 1. $l_1 = p - 1$ *and* $t = 1$.

In this case, $s = 1/(p-1)$ and so $s' = 1$. Then $S(s) = S(u_s) = (p-1, p-1)$ and $S(s') = S(u_{s'}) = (1, 1)$. Let R be the symmetrized subset of $F(a, b)$ generated by the single relator $u_{1/p}$ of the upper presentation of $G(K(1/p))$. By using the fact that $S(1/p) = (p, p)$ and Lemma 3.5, we can make a reduced annular R-diagram M consisting of one 2-cell as depicted in Figure 6 with the following properties:

(i) $\phi(\alpha)$ is an alternating word with initial letter a;

(ii) the S-sequence of $\phi(\alpha)$ is $(p-1, p-1)$.

Here, α denotes the outer boundary cycle of M starting from v_0, where v_0 is the vertex lying in both the outer and inner boundaries of M. It then follows from Lemma 3.5(1) that $\phi(\alpha) \equiv u_s$. Also, if δ denotes the inner boundary cycle of M starting from v_0, then $\phi(\delta)$ is an alternating word with initial letter a^{-1}, and its associated S-sequence is $(1, 1)$. Here, recall from Convention 4.6 that α is read clockwise and δ is read counterclockwise. Then by Lemma 3.5(2), $\phi(\delta)$ is a cyclic permutation of $u_{s'}^{\pm 1}$. Thus it follows from Lemma 4.7 that u_s and $u_{s'}^{\pm 1}$ are conjugate in $G(K(1/p))$.

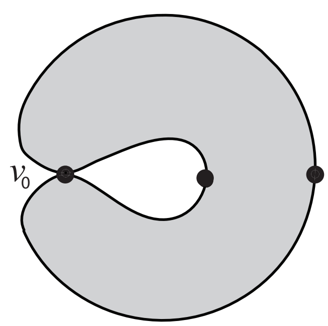


Figure 6. An annular R-diagram M consisting of one 2-cell

Case 2. $0 < l_1 \leq p - 2$ *and* $t \geq 1$.

By Lemma 3.6(1), the sequence $S(s)$ has length $2q_1$. Let $s_j(s)$ denote the j-th term of $S(s)$ for every integer $1 \leq j \leq 2q_1$. By Lemma 3.6(2), $s_j(s) =$

$s_{q_1+j}(s)$ for every $1 \le j \le q_1$. So $S(s) = (s_1(s), \ldots, s_{q_1}(s), s_1(s), \ldots, s_{q_1}(s))$. Here, since $0 \ne s \in I_1 \cup I_2$, each $s_j(s) < p$ by Lemma 3.10(2). For simplicity of notation, put

$$a_1 := s_1(s) - 1 \quad \text{and} \quad a_j := s_j(s) \quad \text{for every } 2 \le j \le q_1.$$

Then the sequence $(a_1, a_2, \ldots, a_{q_1}) = S(\hat{u}_s)$ is symmetric by Claim in the proof of Lemma 5.3 in [4].

On the other hand, if $s_j(s')$ denotes the j-th term of $S(s')$, then again by Lemma 3.6(2)

$$S(s') = (s_1(s'), \ldots, s_{q_1}(s'), s_1(s'), \ldots, s_{q_1}(s')).$$

We now recall a few notation. For a real number t, let $\lfloor t \rfloor$ be the greatest integer not exceeding t, $\lfloor t \rfloor_*$ the greatest integer smaller than t, and $\lceil t \rceil^*$ be the smallest integer greater than t. Then, $\lfloor t \rfloor_* = \lfloor t \rfloor < \lceil t \rceil^*$ for a non-integral real number t, whereas $n - 1 = \lfloor n \rfloor_* < \lfloor n \rfloor < \lceil n \rceil^* = n + 1$ for an integer n. In particular, $\lceil t \rceil^*$ is equal to $\lfloor t \rfloor_* + 1$ or $\lfloor t \rfloor_* + 2$ according as $t \notin \mathbb{Z}$ or $t \in \mathbb{Z}$. We also note

$$\lfloor -t \rfloor_* = -\lceil t \rceil^*, \quad \lfloor t + n \rfloor_* = \lfloor t \rfloor_* + n$$

for every $t \in \mathbb{R}$ and $n \in \mathbb{Z}$.

By Lemma 3.6(1), for every $1 \le j \le q_1$,

$$\begin{aligned} s_j(s') &= s_j(q_1/(pq_1 - p_1)) \\ &= \lfloor j(pq_1 - p_1)/q_1 \rfloor_* - \lfloor (j-1)(pq_1 - p_1)/q_1 \rfloor_* \\ &= \lfloor jp - jp_1/q_1 \rfloor_* - \lfloor (j-1)p - (j-1)p_1/q_1 \rfloor_* \\ &= p + \lfloor -jp_1/q_1 \rfloor_* - \lfloor -(j-1)p_1/q_1 \rfloor_* \\ &= p - \lceil jp_1/q_1 \rceil^* + \lceil (j-1)p_1/q_1 \rceil^*. \end{aligned}$$

For every j with $1 < j < q_1$,

$$\begin{aligned} s_j(s') &= p - (\lfloor jp_1/q_1 \rfloor_* + 1) + (\lfloor (j-1)p_1/q_1 \rfloor_* + 1) \\ &= p - (\lfloor jp_1/q_1 \rfloor_* - \lfloor (j-1)p_1/q_1 \rfloor_*) \\ &= p - s_j(s) = p - a_j. \end{aligned}$$

For $j = 1$,

$$\begin{aligned} s_1(s') &= p - \lceil p_1/q_1 \rceil^* + \lceil 0 \rceil^* \\ &= p - (\lfloor p_1/q_1 \rfloor_* + 1) + 1 \\ &= p - \lfloor p_1/q_1 \rfloor_* \\ &= p - \lfloor p_1/q_1 \rfloor_* + \lfloor 0 \rfloor_* + 1 \\ &= p - s_1(s) + 1 = p - a_1 \end{aligned}$$

For $j = q_1$,

$$\begin{aligned}
s_{q_1}(s') &= p - \lceil p_1 \rceil^* + \lceil (q_1-1)p_1/q_1 \rceil^* \\
&= p - (\lfloor p_1 \rfloor_* + 2) + (\lfloor (q_1-1)p_1/q_1 \rfloor_* + 1) \\
&= p - 1 - (\lfloor p_1 \rfloor_* - \lfloor (q_1-1)p_1/q_1 \rfloor_*) \\
&= p - 1 - s_{q_1}(s) = p - a_{q_1} - 1.
\end{aligned}$$

It then follows that

$$S(s') = (p-a_1, p-a_2, \dots, p-a_{q_1}-1, p-a_1, p-a_2, \dots, p-a_{q_1}-1).$$

Here, recall that $(a_1, a_2, \dots, a_{q_1}) = S(\hat{u}_r)$ is symmetric. So

$$S(s') = (p-a_{q_1}, \dots, p-a_2, p-a_1-1, p-a_{q_1}, \dots, p-a_2, p-a_1-1).$$

Then

$$CS(s') = ((p-a_{q_1}, \dots, p-a_2, p-a_1-1, p-a_{q_1}, \dots, p-a_2, p-a_1-1)).$$

Here, by Corollary 3.9, $CS(s')$ is symmetric. Thus we finally have

$$\begin{aligned}
CS(s') &= ((p-a_1-1, p-a_2, \dots, p-a_{q_1}, p-a_1-1, p-a_2, \dots, p-a_{q_1})) \\
&= ((p-s_1(s), \dots, p-s_{q_1}(s), p-s_1(s), \dots, p-s_{q_1}(s))).
\end{aligned}$$

By using the fact that $S(1/p) = (p,p)$ and Lemma 3.5, we make a reduced annular R-diagram M consisting of q_1 2-cells as depicted in Figure 5(a) with the following properties:

(i) $\phi(\alpha)$ is an alternating word with initial letter a;
(ii) the S-sequence of $\phi(\alpha)$ is the same as $S(s)$.

Here, α denotes the outer boundary cycle of M starting from v_0, where v_0 is a vertex lying in both the outer and inner boundaries of M. It then follows from Lemma 3.5(1) that $\phi(\alpha) \equiv u_s$. Also, if δ denotes the inner boundary cycle of M starting from v_0, then $\phi(\delta)$ is an alternating word with initial letter a^{-1}, and its associated cyclic S-sequence is the same as $CS(s')$. Then by Lemma 3.5(2), $\phi(\delta)$ is a cyclic permutation of $u_{s'}^{\pm 1}$. Thus it follows from Lemma 4.7 that u_s and $u_{s'}^{\pm 1}$ are conjugate in $G(K(1/p))$. $\square$

6. Proof of the only if part of Main Theorem 2.7

We first note that the special case where $p = 2$ can be settled very easily as follows. In this case, $G(K(1/2)) = \langle a, b \mid u_{1/2} \rangle$ with $u_{1/2} = aba^{-1}b^{-1}$ is the rank 2 free abelian group with basis $\{a, b\}$. On the other hand, $I_1(1/2) \cup I_2(1/2) = \{0, 1\}$, and it is obvious that $u_0 = ab$ is not conjugate to $u_1^{\pm 1} = (ab^{-1})^{\pm 1}$ in

$G(K(1/2))$, so that α_0 is not homotopic to α_1 in $S^3 - K(1/2)$. Hence the only if part of Main Theorem 2.7 is valid when $p = 2$.

Proof of the only if part of Main Theorem 2.7. As noted in the beginning of this section, we may assume that p is a positive integer with $p \geq 3$. For two distinct elements $s, s' \in I_1(1/p) \cup I_2(1/p)$, suppose that the unoriented loops α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(1/p)$. Then u_s and $u_{s'}^{\pm 1}$ are conjugate in $G(K(1/p))$. Let R be the symmetrized subset of $F(a, b)$ generated by the single relator $u_{1/p}$ of the upper presentation of $G(K(1/p))$. Due to Lemma 4.7, there is a reduced annular R-diagram M such that u_s and $u_{s'}^{\pm 1}$ are, respectively, outer and inner boundary labels of M. Then we see that M satisfies the three hypotheses (i), (ii) and (iii) of Theorem 4.9. In fact, hypothesis (iii) follows from Lemma 3.10(1) and Remark 3.11.

Lemma 6.1. *Suppose that $((b_1, b_2, \dots, b_n))$ denotes the cyclic S-sequence of an outer boundary label of M. Then $b_i < p$ and the cyclic S-sequence of an inner boundary label of M is $((p - b_1, p - b_2, \dots, p - b_n))$.*

We complete the proof of the only if part of Main Theorem 2.7, by assuming the above lemma. Note that the cyclic S-sequences of arbitrary outer and inner boundary labels of M are $CS(u_s)$ and $CS(u_{s'}^{\pm 1}) = CS(u_{s'})$, respectively. Putting $CS(u_s) = ((b_1, b_2, \dots, b_n))$, Lemma 6.1 implies that $CS(u_{s'}) = ((p - b_1, p - b_2, \dots, p - b_n))$. In particular, $CS(u_s)$ and $CS(u_{s'})$ have the same length. Since the length of $CS(u_s)$ is even or 1 according as $s \neq 0$ or $s = 0$ (see Lemma 3.6(1) and Remark 3.7) and since s and s' are distinct, we see that s nor s' is 0. Hence we may write $s = q_1/p_1$ and $s' = q_2/p_2$, where (p_i, q_i) is a pair of relatively prime positive integers. Then by Lemma 3.6(1), we have $n = 2q_1 = 2q_2$, so $q_1 = q_2$. Also by Lemma 3.6(1),

$$\sum_{i=1}^{q_1} b_i = \sum_{i=1}^{q_1} s_i(s) = \lfloor p_1 \rfloor_* - \lfloor 0 \rfloor_* = p_1$$

and similarly

$$\sum_{i=1}^{q_1} (p - b_i) = \sum_{i=1}^{q_1} s_i(s') = \lfloor p_2 \rfloor_* - \lfloor 0 \rfloor_* = p_2.$$

Hence $p_2 = \sum_{i=1}^{q_1} (p - b_i) = pq_1 - p_1$, which implies that $q_1/(p_1 + p_2) = 1/p$, as required. $\square$

It remains to prove Lemma 6.1.

Proof of Lemma 6.1. Let J be the outer boundary layer of M. Due to Corollary 4.11, J is depicted as in Figure 5(a). Let α and δ be, respectively, the outer and inner boundary cycles of J starting from v_0, where v_0 is a vertex lying in both the outer and inner boundaries of J. Here, recall from Convention 4.6 that α is read clockwise and δ is read counterclockwise. Let $\alpha = e_1, e_2, \dots, e_{2m}$ and $\delta^{-1} = e'_1, e'_2, \dots, e'_{2m}$ be the decompositions into oriented edges in ∂J. Then clearly for each $j = 1, \dots, m$, there is a face D_j of J such that $e_{2j-1}, e_{2j}, e'^{-1}_{2j}, e'^{-1}_{2j-1}$ are consecutive edges in a boundary cycle of D_j.

Claim 1. *For each 2-cell D_j, there are decompositions $v_2 \equiv v_{2b}v_{2e}$ and $v_4 \equiv v_{4b}v_{4e}$ such that the following hold, where $u_{1/p} \equiv v_2v_4$ as in Lemma 3.15, and v_{ib} and v_{ie} are nonempty initial and terminal subwords of v_i.*

(1) *If the cyclic word $(\phi(\partial D_j))$ is equal to $(u_{1/p})$, then $(\phi(e_{2j-1}), \phi(e_{2j}), \phi(e'^{-1}_{2j}), \phi(e'^{-1}_{2j-1}))$ is equal to $(v_{2e}, v_{4b}, v_{4e}, v_{2b})$ or $(v_{4e}, v_{2b}, v_{2e}, v_{4b})$.*
(2) *If the cyclic word $(\phi(\partial D_j))$ is equal to $(u^{-1}_{1/p})$, then $(\phi(e^{-1}_{2j}), \phi(e^{-1}_{2j-1}), \phi(e'_{2j-1}), \phi(e'_{2j}))$ is equal to $(v_{2e}, v_{4b}, v_{4e}, v_{2b})$ or $(v_{4e}, v_{2b}, v_{2e}, v_{4b})$.*

Proof of Claim 1. We treat the case where the cyclic word $(\phi(\partial D_j))$ is equal to $(u_{1/p})$. (The other case can be treated similarly.) Since $\phi(e_{2j-1})\phi(e_{2j})$ is not a piece by Convention 4.3, it contains a maximal 1-piece as a proper initial subword, v. Assume that v is v^*_{2b} (resp. v^*_{4b}) (see Lemma 3.15(1)). Since $\phi(e_{2j-1})\phi(e_{2j})$ is contained in a maximal 2-piece, this assumption together with Lemma 3.15(2) implies that $\phi(e_{2j-1})\phi(e_{2j})$ is v_2 or v_4, and hence $S(\phi(e_{2j-1})\phi(e_{2j})) = (p)$. But, since $s \in I_1(1/p) \cup I_2(1/p)$, every term of the cyclic word $(u_s) = (\phi(\alpha))$ is at most $p-1$ by Lemma 3.10(2) and by Remark 3.11. This implies that the cyclic word $(u_s) = (\phi(\alpha))$ cannot contain a subword whose S-sequence is (p), a contradiction. Hence, the word v is equal to v_{2e} or v_{4e} by Lemma 3.15(1). This together with Lemma 3.15 implies that $(\phi(e_{2j-1}), \phi(e_{2j})) = (v_{2e}, v_{4b})$ or (v_{4e}, v_{2b}) accordingly. By applying a parallel argument to $(\phi(e'^{-1}_{2j}), \phi(e'^{-1}_{2j-1}))$ and by using the fact that the cyclic word $(\phi(\partial D_j))$ is equal to $(u_{1/p})$, we obtain the desired result. □

By Claim 1, there are positive integers ℓ_i and ℓ'_i, for each $1 \le i \le 2m$, such that $S(\phi(e_i)) = (\ell_i)$ and $S(\phi(e'_i)) = (\ell'_i)$ and such that $\ell_i + \ell'_i = p$. Furthermore, they satisfy the following.

Claim 2. *For each $1 \le j \le m$, the following hold.*

(1) $S(\phi(e_{2j-1})\phi(e_{2j})) = (\ell_{2j-1}, \ell_{2j})$ *and* $S(\phi(e'_{2j-1})\phi(e'_{2j})) = (\ell'_{2j-1}, \ell'_{2j})$.

(2) $S(\phi(e_{2j})\phi(e_{2j+1})) = (\ell_{2j}, \ell_{2j+1})$ *and* $S(\phi(e'_{2j})\phi(e'_{2j+1})) = (\ell'_{2j}, \ell'_{2j+1})$, *where the indices are taken modulo* $2m$.

Proof of Claim 2. The first assertion immediately follows from Claim 1. To see the second assertion, note that Claim 1 implies that if $\phi(e_{2j-1})$ is a positive (resp. negative) word then $\phi(e'_{2j})$ is also a positive (resp. negative) word and $\phi(e'_{2j-1})$ and $\phi(e_{2j})$ are negative (resp. positive) words. In particular, if $\phi(e_i)$ is a positive (resp. negative) word then $\phi(e'_i)$ is a negative (resp. positive) word, i.e., $\phi(e_i)$ and $\phi(e'_i)$ always have "opposite signs". Now consider the S-sequence of $\phi(e_{2j})\phi(e_{2j+1})$. Clearly $S(\phi(e_{2j})\phi(e_{2j+1}))$ is either $(\ell_{2j} + \ell_{2j+1})$ or (ℓ_{2j}, ℓ_{2j+1}). We will show that only the latter is possible.

Suppose on the contrary that $S(\phi(e_{2j})\phi(e_{2j+1}))$ is $(\ell_{2j} + \ell_{2j+1})$ for some j. Then $\phi(e_{2j})$ and $\phi(e_{2j+1})$ share the same sign and hence $\phi(e'_{2j})$ and $\phi(e'_{2j+1})$ share the same sign by the above observation. So $S(\phi(e'_{2j})\phi(e'_{2j+1}))$ is $(\ell'_{2j} + \ell'_{2j+1})$. By Lemma 3.10(2) and Remark 3.11, $(u_s) = (\phi(\alpha))$ cannot contain a subword whose S-sequence is (p). So $\ell_{2j} + \ell_{2j+1} < p$. But then $S(\phi(e'_{2j})\phi(e'_{2j+1}))$ is $(\ell'_{2j} + \ell'_{2j+1}) = (2p - \ell_{2j} - \ell_{2j+1})$ with $2p - \ell_{2j} - \ell_{2j+1} > p$. If $J = M$, this contradicts the assumption that $s' \in I_1(1/p) \cup I_2(1/p)$, because every term of $CS(s') = CS(u_{s'})$ has to be at most $p-1$ again by Lemma 3.10(2) and Remark 3.11. Also if $J \subsetneq M$, then, as depicted in Figure 5(b), e'_{2j} and e'_{2j+1} are two consecutive edges in $\partial D'_j \cap \delta^{-1}$ for some face D'_j in $M - J$, but then $(u_{1/p})$ contains a subword whose S-sequence is $(2p - \ell_{2j} - \ell_{2j+1})$ with $2p - \ell_{2j} - \ell_{2j+1} > p$, contradicting $CS(u_{1/p}) = ((p, p))$. □

Therefore $CS(\phi(\alpha))$ becomes $((\ell_1, \ell_2, \dots, \ell_{2m-1}, \ell_{2m}))$, and $CS(\phi(\delta^{-1}))$ becomes $((p-\ell_1, p-\ell_2, \dots, p-\ell_{2m-1}, p-\ell_{2m}))$. If $M = J$, the proof of Lemma 6.1 is completed. Now assume $J \subsetneq M$. Let J_1 denote the outer boundary layer of $M - int(J)$. Then the cyclic S-sequence of an arbitrary inner boundary label of J_1 is $((\ell_1, \ell_2, \dots, \ell_{2m-1}, \ell_{2m}))$. If $M = J \cup J_1$, then $s = s'$, a contradiction. Hence we must have $J \cup J_1 \subsetneq M$. Let J_2 denote the outer boundary layer of $M - int(J \cup J_1)$. Then the cyclic S-sequence of an arbitrary inner boundary label of J_2 is $((p - \ell_1, p - \ell_2, \dots, p - \ell_{2m-1}, p - \ell_{2m}))$. If $M = J \cup J_1 \cup J_2$, the proof of Lemma 6.1 is completed. By the repetition of this argument, we are done. □

7. Topological proof of the if part of Main Theorem 2.7

Consider the involution h of $(S^3, K(1/p))$ as illustrated in Figure 7. Then h satisfies the following conditions.

(i) The automorphism h_* of $G(K(1/p)) = \langle a, b \,|\, u_{1/p} \rangle$ induced by h is given by $h_*(a) = a^{-1}$ and $h_*(b) = b^{-1}$.
(ii) h interchanges $(B^3, t(\infty))$ and $(B^3, t(1/p))$.

We note that though every 2-bridge link $K(r)$ admits an involution h satisfying the first condition, h can be isotoped so that it also satisfies the second condition only when $K(r)$ is equivalent to $K(\pm 1/p)$ for some $p \in \mathbb{Z}_+$ (see [1] and [10]).

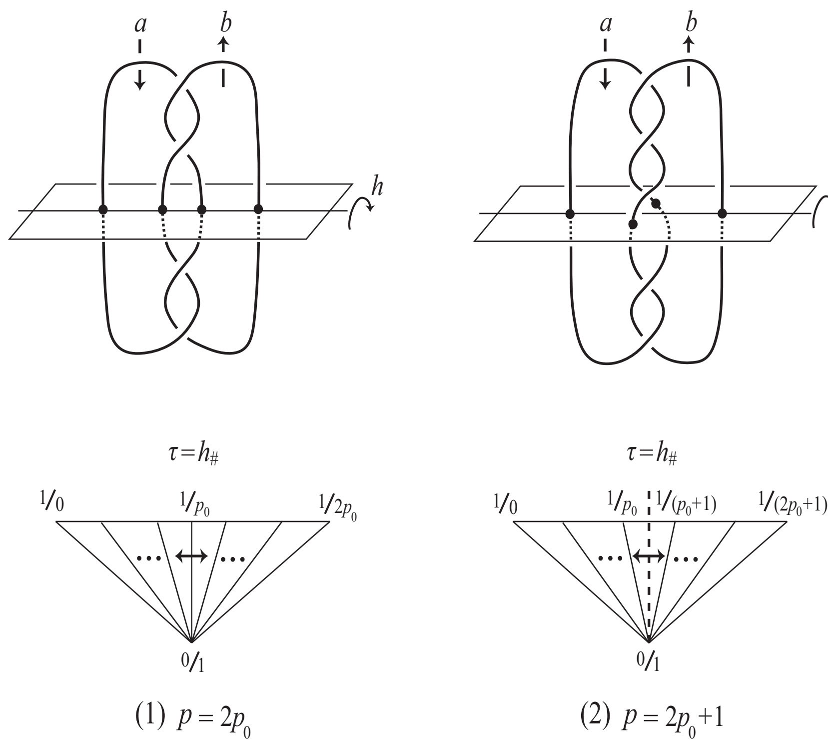


FIGURE 7. The involution h

The restriction of h to the bridge sphere $\boldsymbol{S}$ induces an automorphism, $h_\#$, of the curve complex of $\boldsymbol{S}$, i.e., the Farey tessellation.

Lemma 7.1. *The automorphism $h_\#$ is equal to the automorphism τ defined in the introduction. Moreover, we have $h_\#(c/d) = c/(cp - d)$ for every $c/d \in \hat{\mathbb{Q}}$.*

Proof. Since α_∞ and $\alpha_{1/p}$, respectively, are the unique essential simple loops on $\boldsymbol{S}$ which are null-homotopic in $B^3 - t(\infty)$ and $B^3 - t(1/p)$, the second condition on h implies that $h|_{\boldsymbol{S}}$ interchanges α_∞ and $\alpha_{1/p}$, i.e., $h_\#$ interchanges ∞ and

$1/p$. Let σ_i be the Farey triangle $\langle 0, 1/(i-1), 1/i\rangle$ for $i = 1, \ldots, p$. Then $\{\sigma_1, \ldots, \sigma_p\}$ is equal to the set of Farey triangles whose interiors intersect the hyperbolic geodesic, ℓ, joining ∞ with $1/p$. Since $h_\#$ preserves ℓ, it preserves the set $\{\sigma_1, \ldots, \sigma_p\}$, and hence $h_\#(\sigma_i) = \sigma_{p+1-i}$. This implies that $h_\#$ is equal to the automorphism τ.

Now set $A = \begin{pmatrix} -1 & p \\ 0 & 1 \end{pmatrix}$ and consider the self-homeomorphism of $\boldsymbol{S} = (\mathbb{R}^2 - \mathbb{Z}^2)/H$ induced by the linear transformation $x \mapsto Ax$ of $\mathbb{R}^2 - \mathbb{Z}^2$. Let $A_\#$ be the action of the self-homeomorphism on the Farey tessellation. Then $A_\#$ is equal to $\tau = h_\#$. In fact, since

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad A\begin{pmatrix} p \\ 2 \end{pmatrix} = \begin{pmatrix} p \\ 2 \end{pmatrix}$$

we see that $A_\#$ fixes 0 and $2/p$. Since $\det A = -1$, this implies that $A_\#$ is the reflection in the geodesic joining 0 and $2/p$. Hence $A_\# = \tau = h_\#$. Since $A\begin{pmatrix} d \\ c \end{pmatrix} = \begin{pmatrix} -d+pc \\ c \end{pmatrix}$, we see $h_\#(c/d) = A_\#(c/d) = c/(cp-d)$. $\square$

On the other hand, the following lemma holds for the automorphism h_* of $G(K(1/p))$.

Lemma 7.2. *For every slope $s \in \hat{\mathbb{Q}}$, $h_*(u_s)$ is conjugate to u_s or u_s^{-1} in $G(K(1/p))$.*

Proof. Since $h_*(a) = a^{-1}$ and $h_*(b) = b^{-1}$, $h_*(u_s)$ is represented by an alternating word whose S-sequence is equal to $S(u_s)$. By Lemma 3.5(2), any alternating word v with $S(v) = S(u_s)$ is conjugate to u_s or u_s^{-1} in $G(K(1/p))$. Hence we obtain the desired result. $\square$

Topological proof of the if part of Main Theorem 2.7. By Lemma 7.1, $\alpha_{c/(cp-d)} = h(\alpha_{c/d})$. On the other hand, by Lemma 7.2, $h(\alpha_{c/d})$ is homotopic to $\alpha_{c/d}$ in $S^3 - K(1/p)$. Hence $\alpha_{c/(cp-d)}$ is homotopic to $\alpha_{c/d}$ in $S^3 - K(1/p)$. $\square$

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